Exercice : Différentiabilité du déterminant

$$f \left| \begin{array}{ccc} \mathcal{M}_n(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ A & \longmapsto & \det(A) \end{array} \right. \quad (n \in \mathbb{N}^*)$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma) \cdot \prod_{k=1}^{n} [A]_{k,\sigma(k)}$$

f est polynomiale donc $\mathcal{C}^\infty$.

En I_n : Soit $H \in \mathcal{M}_n(\mathbb{R}) \setminus \{0\}$, $\varepsilon \in \mathbb{R}^*$,

$$df(I_n) \cdot H = \lim_{\varepsilon \xrightarrow[\mathbb{R}]{} 0} \frac{1}{\varepsilon}\left(f(I_n + \varepsilon H) - \det(I_n)\right)$$

$$= \lim_{\varepsilon \xrightarrow[\mathbb{R}]{} 0} \frac{1}{\varepsilon}\left(\varepsilon^n \det\left(\frac{1}{\varepsilon} I_n + H\right) - 1\right)$$

$$= \lim_{\varepsilon \xrightarrow[\mathbb{R}]{} 0} \frac{1}{\varepsilon}\left(\varepsilon^n \chi_{-H}\left(\frac{1}{\varepsilon}\right) - 1\right)$$

$$= \lim_{\varepsilon \xrightarrow[\mathbb{R}]{} 0} \frac{1}{\varepsilon}\left(\sum_{k=0}^{n} [\chi_{-H}]_k \, \varepsilon^{n-k} - 1\right)$$

$$= [\chi_{-H}]_{n-1}$$

$$= -\mathrm{Tr}(-H)$$

$$= \mathrm{Tr}(H)$$

En $A \in Gl_n(\mathbb{R})$: Soit $H \in \mathcal{M}_n(\mathbb{R}) \setminus \{0\}$,

$$df(A) \cdot H = \lim_{\substack{\varepsilon \to 0 \\ \mathbb{R}}} \frac{1}{\varepsilon}\left(\det(A + \varepsilon H) - \det(A)\right)$$

$$= \lim_{\substack{\varepsilon \to 0 \\ \mathbb{R}}} \frac{1}{\varepsilon} \det(A)\left(\det(I_n + \varepsilon A^{-1}H) - \det(I_n)\right)$$

$$= \det(A)\, df(I_n) \cdot (A^{-1}H)$$

$$= \det(A)\, \operatorname{Tr}(A^{-1}H)$$

$$= \operatorname{Tr}(\operatorname{Com}(A)^T H)$$

$$= \langle \operatorname{Com}(A), H \rangle$$

Donc $\forall A \in Gl_n(\mathbb{R})$, $df(A) = \langle \operatorname{Com}(A), \cdot \rangle$.

On conclut par prolongement par continuité et densité.

Exercice 87 : 2. $c \in \mathbb{R}^*$, $f \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$,

f vérifie $(\mathcal{E})$:

$$\forall (x, t) \in \mathbb{R}^2, \quad c^2 \frac{\partial^2 f}{\partial x^2}(x, t) = \frac{\partial^2 f}{\partial t^2}(x, t)$$

Soient $\alpha, \beta, \gamma, \delta \in \mathbb{R}$,

$$g \left| \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, t) & \longmapsto & f(\alpha x + \beta t, \gamma x + \delta t) \end{array} \right.$$

a) Montrons que g est $\mathcal{C}^2$ sur $\mathbb{R}^2$:

On pose $h \left| \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (u, v) & \longmapsto & (\alpha u + \beta v, \gamma u + \delta t) = (h_1(u, v), h_2(u, v)) \end{array} \right.$.

On a donc : $g = f \circ h$.

h est $\mathcal{C}^2$ car linéaire, donc g est $\mathcal{C}^2$ comme composée d'applications $\mathcal{C}^2$:

Soient $(u, v) \in \mathbb{R}^2$,

$$\frac{\partial g}{\partial u}(u, v) = \frac{\partial f}{\partial x}(\overbrace{\alpha u + \beta v, \gamma u + \delta v}^{*}) \times \alpha + \frac{\partial f}{\partial t}(*) \times \gamma$$

$$\cancel{\frac{\partial}{\partial v}\left(\frac{\partial g}{\partial u}\right)(u, v) = \frac{\partial^2 f}{\partial x^2}(*) \times \beta \times \alpha + \frac{\partial^2 f}{\partial x \partial t}(*) \times \beta \times + \frac{\partial^2 f}{\partial t \partial x}(*) \times \delta \times \alpha}$$

$$\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right)(u,v) = \frac{\partial^2 f}{\partial x^2}(*) \times \beta \times \alpha$$
$$+ \frac{\partial^2 f}{\partial t \partial x}(*) \times \delta \times \alpha$$
$$+ \frac{\partial^2 f}{\partial x \partial t}(*) \times \beta \times \gamma$$
$$+ \frac{\partial^2 f}{\partial t^2}(*) \times \delta \times \gamma$$

(Remarque : $\alpha = \frac{\partial h_1}{\partial u}(u,v)$

$\beta = \frac{\partial h_1}{\partial v}(u,v)$

$\gamma = \frac{\partial h_2}{\partial u}(u,v)$

$\delta = \frac{\partial h_2}{\partial v}(u,v)$)

b) (i) : " $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ est inversible "

(ii) : " $\forall (u,v) \in \mathbb{R}^2, \ \frac{\partial^2 g}{\partial v \partial u}(u,v) = 0$ "

On sait que : $\forall (u,v) \in \mathbb{R}^2$,

$$\frac{\partial^2 g}{\partial v \partial u}(u,v) = \frac{\partial^2 f}{\partial x^2}(*)\beta\alpha + \frac{\partial^2 f}{\partial t^2}(*)\delta\gamma + \frac{\partial^2 f}{\partial t \partial x}(\beta\gamma + \delta\alpha)$$

(Théorème de Schwarz)

$$= \frac{\partial^2 f}{\partial x^2}(*)(\beta\alpha + c^2\delta\gamma) + \frac{\partial^2 f}{\partial t \partial x}(*)(\beta\gamma + \delta\alpha)$$

(f vérifie $(\mathcal{E})$)

Afin de vérifier (ii), il suffit que :

$$\beta\alpha + c^2\delta\gamma = 0$$
et $\beta\gamma + \delta\alpha = 0$

Par ailleurs, (i) $\iff \alpha\delta - \gamma\beta \neq 0$.

On pose alors $\beta = \alpha = c$,
et obtient $c^2 + c^2\delta\gamma = 0$,

puis $\gamma = -1$, $\delta = 1$.

De (ii), on déduit :

$\exists (\varphi, \psi) \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})^2$,

$\forall (u, v) \in \mathbb{R}^2$, $g(u, v) = \varphi(u) + \psi(v)$

(Résultat de la question 1.)

donc $f(\alpha u + \beta v, \gamma u + \delta v) = \varphi(u) + \psi(v)$.

Soient $(x, t) \in \mathbb{R}^2$,

$$\begin{cases} x = \alpha u + \beta v \\ t = \gamma u + \delta v \end{cases} \iff \begin{cases} x = cu + cv \\ t = -u + v \end{cases}$$

$$\iff \begin{cases} x + ct = 2cv & (L_1 \leftarrow L_1 + cL_2) \\ t = -u + v \end{cases}$$

$$\iff \begin{cases} v = \dfrac{x}{2c} + \dfrac{t}{2} \\ u = \dfrac{t}{2} - \dfrac{x}{2c} \end{cases}$$

Donc $f(x, t) = \varphi\left(\dfrac{t}{2} - \dfrac{x}{2c}\right) + \psi\left(\dfrac{t}{2} + \dfrac{x}{2c}\right)$.

Pour finir, on pose :

$$\tilde{\varphi} : y \longmapsto \varphi\left(\frac{y}{2c}\right)$$

$$\tilde{\psi} : x \longmapsto \varphi\left(-\frac{x}{2c}\right)$$

pour écrire : $f(x,t) = \tilde{\varphi}(x+ct) + \tilde{\psi}(x-ct)$

Exercice 12 :

1. Soit $\gamma \,\Big|\, \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R}^2 \\ t \longmapsto (\gamma_1(t), \gamma_2(t)) \\ \qquad = (x+t, y+t) \end{array}$

où $(x,y) \in \mathbb{R}^2$ fixés.

γ est dérivable car γ_1 et γ_2 le sont (elles sont affines).

Par dérivation le long d'un chemin, $\tau = f \circ \gamma$ est dérivable :

$$\forall t \in \mathbb{R}, \ (f \circ \gamma)'(t) = df(\gamma(t)) \cdot \gamma'(t)$$

$$= df(\gamma(t)) \cdot (\gamma_1'(t), \gamma_2'(t))$$

$$= \frac{\partial f}{\partial x}(x+t, y+t) + \frac{\partial f}{\partial y}(x+t, y+t)$$

$$\left(= \langle \nabla f(\gamma(t)), \gamma'(t) \rangle \right)$$

2. a) Soit $(x,y) \in \mathbb{R}^2$,

on a $\forall \varepsilon \in \mathbb{R}$, $f(x+\varepsilon, y+\varepsilon) = f(x,y)$

donc $\tau(\varepsilon) = \tau(0)$

donc $\tau'(\varepsilon) = 0$

et $\tau(\varepsilon) = \frac{\partial f}{\partial x}(x+\varepsilon, y+\varepsilon) + \frac{\partial f}{\partial y}(x+\varepsilon, y+\varepsilon)$

En particulier, $\frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y) = 0$

b) $h \left| \begin{array}{l} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ (u,v) \longmapsto (au+bv, cu+dv) = (h_1(u,v), h_2(u,v)) \end{array} \right.$

est différentiable sur $\mathbb{R}^2$,

donc $g = f \circ h$ l'est aussi.

Donc $\forall (u,v) \in \mathbb{R}^2$, $\frac{\partial g}{\partial v}(u,v)$ existe :

$$\frac{\partial g}{\partial v}(u,v) = \frac{\partial f}{\partial x}(*) \cdot \frac{\partial h_1}{\partial v}(u,v) + \frac{\partial f}{\partial y}(*) \cdot \frac{\partial h_2}{\partial v}(u,v)$$

où $* = (au+bv, cu+dv)$.

c) On pose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$\forall (u,v) \in \mathbb{R}^2, \ \frac{\partial g}{\partial v}(u,v) = 0$.

Soit $u \in \mathbb{R}$,

$$0 = \int_0^b \frac{\partial g}{\partial v}(u,v)\,dv$$

$$= g(u,b) - g(u,0)$$

donc $\forall (u,v) \in \mathbb{R}^2, \ g(u,v) = g(u,0)$

donc $f(u+v, v) = g(u,0)$.

Posons $\begin{cases} x = u + v \\ y = v \end{cases}$

donc $\begin{cases} u = x - y \\ v = y \end{cases}$

donc $f(x,y) = g(x-y, 0)$

$\varphi : z \longmapsto g(z,0)$

$(z \mapsto (z,0)$ linéaire $)$