TD Correction

Exercice 87 (cours)

1) Si $g \left| \begin{array}{l} \mathbb{R}^2 \to \mathbb{R} \\ (u,v) \mapsto g(u,v) \end{array} \right.$ $\mathcal{C}^2$ vérifie $\forall (u,v) \in \mathbb{R}^2, \dfrac{\partial^2 g}{\partial u \partial v}(u,v) = 0$

Alors $\exists \varphi, \psi \in \mathcal{C}^2(\mathbb{R},\mathbb{R}), \forall (u,v) \in \mathbb{R}^2, g(u,v) = \varphi(u) + \psi(v)$

2) Soit $c \in \mathbb{R}^*$, $f \left| \begin{array}{l} \mathbb{R}^2 \to \mathbb{R} \\ (x,t) \mapsto f(x,t) \end{array} \right.$ tq f $\mathcal{C}^2$ et

$$\forall (x,t) \in \mathbb{R}^2, \quad c^2 \frac{\partial^2 f}{\partial x^2}(x,t) = \frac{\partial^2 f}{\partial t^2}(x,t)$$

Mq $\exists \varphi, \psi \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$ tq $\forall (x,t) \in \mathbb{R}^2, f(x,t) = \varphi(x+ct) + \psi(x-ct)$

a. $g = f \circ h$ avec $h \left| \begin{array}{l} \mathbb{R}^2 \to \mathbb{R}^2 \\ (u,v) \mapsto \underbrace{(\underbrace{\alpha u + \beta v}_{h_1(u,v)}, \underbrace{\gamma u + \delta v}_{h_2(u,v)})}_{(*)} \end{array} \right.$

f $\mathcal{C}^2$, h lin donc $\mathcal{C}^2$.

- Par règle de la chaîne, soit $(u,v) \in \mathbb{R}^2$,

$$\frac{\partial g}{\partial u}(u,v) = \frac{\partial f}{\partial x}\left(h_1(u,v), h_2(u,v)\right) \times \frac{\partial h_1}{\partial u}(u,v)$$

$$+ \frac{\partial f}{\partial y}\left(h_1(u,v), h_2(u,v)\right) \times \frac{\partial h_2}{\partial u}(u,v)$$

$$= \frac{\partial f}{\partial x}(*)\alpha + \frac{\partial f}{\partial y}(*)\gamma.$$

$$\frac{\partial^2 g}{\partial v \partial u}(u,v) = \alpha\left(\frac{\partial^2 f}{\partial x^2}(*)\frac{\partial h_1}{\partial v}(u,v) + \underline{\frac{\partial^2 f}{\partial y \partial x}}(*)\frac{\partial h_2}{\partial v}(u,v)\right)$$

$$+\gamma\left(\underline{\frac{\partial^2 f}{\partial x \partial y}}(*)\frac{\partial h_1}{\partial v}(u,v) + \underbrace{\frac{\partial^2 f}{\partial y^2}(*)}_{= c^2 \frac{\partial^2 f}{\partial x^2}(*)}\frac{\partial h_2}{\partial v}(u,v)\right)$$

thm Schwartz

$$= (\alpha\beta + c^2\gamma\delta)\frac{\partial^2 f}{\partial x^2}(*) + (\alpha\delta + \gamma\beta)\frac{\partial^2 f}{\partial x \partial y}(*)$$

b) Posons $\alpha = \beta = c$, $\gamma = -1$, $\delta = 1$. $\begin{pmatrix} c & c \\ -1 & 1 \end{pmatrix}$

Alors $\forall (u,v) \in \mathbb{R}^2$, $\frac{\partial^2 g}{\partial v \partial u}(u,v) = 0$

$\exists \varphi, \psi \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$, $\forall (u,v) \in \mathbb{R}^2$, $g(u,v) = \varphi(u) + \psi(v)$

(Q1) $f(cu+cv, -u+v) = \varphi(u) + \psi(v)$

$$\begin{cases} x = cu + cv \\ t = -u+v \end{cases} \underset{c \neq 0}{\iff} \begin{cases} \frac{x}{c} = u+v & (L_1) \\ t = -u+v & (L_2) \end{cases}$$

$(L_1)+(L_2)$ donne $\frac{x}{2c} + \frac{t}{2} = v$

Dans (L_2), $\frac{x}{2c} - \frac{t}{2} = u$

$\exists \varphi, \psi \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$, $\forall (x,t) \in \mathbb{R}^2$, $f(x,t) = \varphi\left(\frac{x}{2c} + \frac{t}{2}\right) + \psi\left(\frac{x}{2c} - \frac{t}{2}\right)$

$\tilde{\varphi} : y \overset{\mathcal{C}^2}{\mapsto} \varphi\left(\frac{y}{2c}\right) \quad \tilde{\varphi}(x+ct) = \varphi\left(\frac{x}{2c} + \frac{t}{2}\right)$

$\tilde{\psi} : y \overset{\mathcal{C}^2}{\mapsto} \psi\left(\frac{y}{2c}\right) \quad \tilde{\psi}(x-ct) = \psi\left(\frac{x}{2c} - \frac{t}{2}\right)$

Exercice 4 TD

$f \left| \begin{array}{l} \mathbb{R}^2 \to \mathbb{R} \\ (x,y) \mapsto x^3 + xy + y^2 \end{array} \right.$

1) Mq f diff sur $\mathbb{R}^2$

2) Calculer $\forall (a,b) \in \mathbb{R}^2, \forall (h_1, h_2) \in \mathbb{R}^2, df(a,b).(h_1, h_2)$

1) Soit $y \in \mathbb{R}$ fixé.

$f(\cdot, y) \left| \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto f(x,y) \end{array} \right.$ est dérivable (polynomial) sur $\mathbb{R}$, et $\forall x \in \mathbb{R}$, $f(\cdot,y)'(x) = 3x^2 + y$. D'où ~~$\forall x \in \mathbb{R}$~~ $\forall (x,y) \in \mathbb{R}^2$, $\frac{\partial f}{\partial x}(x,y)$ existe et vaut $3x^2+y$

De même, $\forall (x,y) \in \mathbb{R}^2$, $\frac{\partial f}{\partial y}(x,y)$ existe et vaut $x+2y$.

$\frac{\partial f}{\partial x} \left| \begin{array}{l} \mathbb{R}^2 \to \mathbb{R} \\ (x,y) \mapsto 3x^2+y \end{array} \right.$ et $\frac{\partial f}{\partial y} \left| \begin{array}{l} \mathbb{R}^2 \to \mathbb{R} \\ (x,y) \mapsto x+2y \end{array} \right.$

Sont continues sur $\mathbb{R}^2$ (polynomiales), donc f est diff par critère $\mathcal{C}^1$

2) Soient $(a,b) \in \mathbb{R}^2$, $(h_1, h_2) \in \mathbb{R}^2$.

$df(a,b).(h_1,h_2) = \langle \nabla f(a,b), (h_1,h_2) \rangle$

$= \frac{\partial f}{\partial x}(a,b) h_1 + \frac{\partial f}{\partial y}(a,b) h_2$

$\boxed{= (3a^2+b)h_1 + (a+2b)h_2.}$

Exercice 11

1) Soit $(x,y) \in \mathbb{R}^2$. Mq g est dér et calculons sa dérivée.

g est une fct° de la variable réelle donc elle est dérivable ssi elle est diff.

$h \mid \begin{array}{l} \mathbb{R} \to \mathbb{R}^2 \\ t \mapsto (tx, ty) \end{array}$, on a $g = f \circ h$ avec h linéaire, donc diff.

$\hookrightarrow dh \mid \begin{array}{l} \mathbb{R} \to \mathcal{L}(\mathbb{R}^2, \mathbb{R}) \\ t \mapsto h. \end{array}$

g est donc dérivable car diff (composée de diff)

$\forall t \in \mathbb{R}, \forall a \in \mathbb{R}, dg(t).a = a g'(t)$
avec $dg(t) = d(f \circ h)(t) = df(h(t)) \circ dh(t)$

Ainsi, $g'(t) = dg(t).1 = df(h(t)) \circ h.1$
$= df(xt, yt).(x,y)$

$= \langle \nabla f(xt, yt), (x,y) \rangle$

avec $\nabla f(xt, yt) = \left(\frac{\partial f}{\partial x}(xt, yt), \frac{\partial f}{\partial y}(xt, yt) \right)$

$$= \frac{\partial f}{\partial x}(xt, yt) . x + \frac{\partial f}{\partial y}(xt, yt) . y$$